DYNAMICAL THEORY OF THE GLOBULAR CLUSTERS AND OF THE CLUSTERING POWER INFERRED BY HERSCHEL FROM THE OBSERVED FIGURES OF SIDEREAL SYSTEMS OF HIGH ORDER.

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(Plates VIII (bis) and IX.)

I. Introductory Remarks.

More than a century and a quarter have elapsed since it was confidently announced by Sir William Herschel that sidereal systems made up of thousands of stars exhibit the effects of a clustering power which is everywhere moulding these systems into symmetrical figures, as if by the continued action of central forces (*Phil. Trans.*, 1785, p. 255, and 1789, pp. 218–226). In support of this view he cited especially the figures of the planetary nebulæ, and the globular clusters, as well as the more expanded and irregular swarms and clouds of stars visible to the naked eye along the course of the Milky Way, which thus appears to traverse the heavens as a *clustering stream.* And yet notwithstanding the early date of this announcement and the unrivaled eminence of Herschel, it is only very recently that astronomers have begun to consider the origin of sidereal systems of the highest order.

The historical difficulty of solving the problem of n-bodies, when n exceeds 2, which dates from the establishment of the law of universal gravitation by Newton in 1687, will sufficiently account for the restriction of the researches of mathematicians to the planetary system, where the central masses always are very predominant, the orbits almost circular and nearly in a common plane, and to other simple systems such as the double and multiple stars: but owing to the general prevalence of the clustering tendency pointed out by Herschel and now found to be at work throughout the sidereal universe, it becomes necessary for the modern investigator to consider also the higher orders of sidereal systems, including those made up of thousands and even millions of stars. It is only by such a comprehensive view of nature, which embraces and unites all types of systems under one common principle, that we may hope to establish the most general laws governing the evolution of the sidereal universe.

Accordingly, although the strict mathematical treatment of the great historical problem of n-bodies is but little advanced by the recent researches of geometers, yet if we could arrive at the general secular tendency in nature, from the observational study of the phenomena presented by highly complex systems of stars, operating under known laws of attractive and repulsive forces, the former for gathering the matter into large masses, the latter for redistributing it in the form of fine dust, the result of such an investigation would guide us towards a grasp of problems too complex for rigorous treatment by any known method of analysis.

Now it happens that in the second volume of the "Researches on the Evolution of the Stellar System," 1910, the writer was able to establish great generality in the processes of cosmogony, and to show that the universal tendency in nature is for the large bodies to drift towards the most powerful centers of attraction, while the only throwing off of masses that ever takes place is that of small particles expelled from the stars under the action of repulsive forces and driven away for the formation of new nebulæ. The repulsive forces thus operate to counteract the clustering tendency noticed by the elder Herschel, and so clearly foreseen by Newton as an inevitable effect of universal gravitation upon the motions of the solar system that he believed the intervention of the Deity eventually would become necessary for the restoration of the order of the world (cf. Newton's "Letters to Bentley," Brewster's "Life of Newton," Vol. II., and Chapter XVII., and Appendix X).

But whilst the argument developed in the second volume of my "Researches" gives unexpected simplicity, uniformity and continuity to the processes of cosmogony, there has not yet been developed, so far as I know, any precise investigation of the attractive forces operating in globular clusters, which might disclose the nature of the clustering power noticed by Herschel to be in progress throughout the sidereal universe. Such an investigation of the central forces governing the motions in clusters is very desirable, because it might be expected to throw light on the mode of evolution of clusters as the highest type of the perfect sidereal system. If it can be shown that a clustering power is really at work, and is of such a nature as to produce these globular masses of stars, it will be less important to consider the details of those systems which have not yet reached a state of symmetry and full maturity; for the governing principle being established for the most perfect types, it must be held to be the same in all.

II. General Expressions for the Potential of an Attracting Mass.

If we have a mass M' of any figure whatever, in which the law of density is $\sigma' = f(x', y', z')$, where (x', y', z') are the coördinates of the element dm' of the attracting mass, and this element attracts a unit mass whose coördinates are (x, y, z); then the element of the attracting mass is

$$dm' = \sigma' dx' dy' dz'. \qquad (1)$$

And the expressions for the forces acting on the unit mass when resolved along the coördinate axes become

$$\begin{aligned} \frac{\partial U}{\partial x} &= X = \int\int\int \frac{x' - x}{r^3} \sigma' dx' dy' dz', \\ \frac{\partial U}{\partial y} &= Y = \int\int\int \frac{y' - y}{r^3} \sigma' dx' dy' dz', \\ \frac{\partial U}{\partial z} &= Z = \int\int\int \frac{z' - z}{r^3} \sigma' dx' dy' dz'. \end{aligned} \qquad (2)$$

$$r = \sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}.$$

The potential function itself obviously is

$$U = \int\int\int \frac{\sigma' dx' dy' dz'}{r}. \qquad (3)$$

In spherical coördinates we may take the angle ϕ for the longitude, θ for the latitude, and r for the radius of the sphere; and then the required expressions become

$$\begin{aligned} x' - x &= r \sin\theta \cos\phi, \\ y' - y &= r \sin\theta \sin\phi, \\ z' - z &= r \cos\theta. \end{aligned} \tag{4}$$

The element of mass dm' defined in (1) has the equivalent form

$$\sigma' dx' dy' dz' = \sigma' dr \cdot r d\theta \cdot r \sin\theta d\phi. \tag{5}$$

The element of the potential due to this differential element is

$$\frac{\sigma' r^2 \sin\theta dr d\theta d\phi}{r}; \tag{6}$$

and the general expression for the potential becomes

$$U = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{r} \sigma' r dr. \tag{7}$$

If we make use of the equations (1), (4), (5) in equation (2) we may obtain the corresponding expressions for the forces resolved along the coördinate axes:

$$\begin{aligned} X &= \int_0^{2\pi} \cos\phi d\phi \int_0^{\pi} \sin^2\theta d\theta \int_0^{r} \sigma' dr, \\ Y &= \int_0^{2\pi} \sin\phi d\phi \int_0^{\pi} \sin^2\theta d\theta \int_0^{r} \sigma' dr, \\ Z &= \int_0^{2\pi} d\phi \int_0^{\pi} \cos\theta \sin\theta d\theta \int_0^{r} \sigma' dr. \end{aligned} \tag{8}$$

These expressions will hold rigorously true for any law of density whatever, so long as it is finite and continuous. In the physical universe these conditions always are fulfilled; and hence if these several integrals can be evaluated, they will give the potentials and forces exerted on a unit mass by an attracting body such as a cluster of stars, or the spherical shell surrounding the nucleus of a cluster.

But before considering the attraction of a cluster in detail, we shall first examine the cumulative effect of central forces on the law of density. The problem is intricate and must be treated by methods of great generality, but as it will elucidate the subsequent procedure for determining the attraction of such a mass upon a neighboring point, we shall give the analysis with enough detail to establish clearly the secular effect of close appulses of individual stars upon the figure and internal arrangement of these wonderful masses of stars.

III. The Cumulative Effect of the Central Forces upon the Figure and Compression of a Globular Cluster of Stars.

Suppose a globular cluster of stars to be in a moderate state of compression, with density increasing towards the center. Imagine the whole of the mass at the epoch t_0 to be divided into two parts by a spherical surface of radius r, drawn about the center of gravity of the entire system; and let the external boundary of the cluster be R, so chosen that no star, from the motions existing at the initial epoch, will cross the border $r=R$. The stars in the outer shell, between the surfaces r and R, with coördinates (x', y', z'), will give rise to a potential U. Those of the nucleus or series of internal shells, between $r=0$, and $r=r$, with coördinates (x, y, z), will give rise to a potential V. Accordingly we have

$$U=\iiint \frac{\sigma' dx' dy' dz'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}},$$
$$V=\iiint \frac{\sigma dx dy dz}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}. \tag{9}$$

And the forces resolved along the coördinate axes are

$$\frac{\partial U}{\partial x}=X=\iiint \frac{\sigma'(x'-x)dx' dy' dz'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}},$$
$$\frac{\partial U}{\partial y}=Y=\iiint \frac{\sigma'(y'-y)dx' dy' dz'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}}, \tag{10}$$
$$\frac{\partial U}{\partial z}=Z=\iiint \frac{\sigma'(z'-z)dx' dy' dz'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}};$$

with similar expressions for

$$\frac{\partial V}{\partial x}, \quad \frac{\partial V}{\partial y}, \quad \frac{\partial V}{\partial z}.$$

The integration for the mutual potential energy of the stars in the outer shell relative to those in the central sphere of radius r leads to a sextuple integral

$$\Omega = \int\int\int\int\int\int \frac{\sigma\sigma' dxdydzdx'dy'dz'}{\sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}}. \quad (11)$$

And the total of the mutual attractive forces resolved along the coördinate axes are

$$\frac{\partial \Omega}{\partial x} = \int\int\int\int\int\int \frac{\sigma\sigma'(x'-x)dxdydzdx'dy'dz'}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{\frac{3}{2}}}, \quad (12)$$

with similar expressions for

$$\frac{\partial \Omega}{\partial y}, \quad \frac{\partial \Omega}{\partial z}.$$

Now it is easy to prove (cf. Thomson and Tait's "Natural Philosophy," §§ 547–548) that the sextuple integral (11) can be put into the form

$$\Omega = \int\int\int \sigma U dxdydz = \int\int\int \sigma' V dx'dy'dz'. \quad (13)$$

By actual derivation of the expressions (9) we easily find that

$$\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial V}{\partial z}$$

is equivalent to Ω, by (11), and therefore

$$\int\int\int \left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial V}{\partial z} \right) dxdydz = 4\pi\Omega, \quad (14)$$

4π being introduced owing to the integration over the closed sphere surface (cf. Williamson's "Integral Calculus," edition of 1896, p. 330; Bertrand, "Calcul Integral," p. 480).

As the right members of (13) give the mutual potential energy of the bodies of the system, it suffices for us to deal with the integral of (14). This triple integral admits of transformation by Green's theorem ("Essay on the Application of Mathematics to Electricity and Magnetism," Nottingham, 1828). If U and V be functions of $x, y, z,$ the rectangular coördinates of a point; then provided U and V are *finite and continuous for all points within a given closed surface* S, it is easy to show (cf. Williamson's "Integral Calculus," 7th edition, 1896, p. 328; Riemann, "Schwere, Electricität und Magnetismus," p. 73; Thomson and Tait's "Natural Philosophy," Part I., Vol., I., p. 167; Bertrand, "Calcul Integral," p. 480):

$$\int\int\int\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\frac{\partial V}{\partial y}+\frac{\partial U}{\partial z}\frac{\partial V}{\partial z}\right)dxdydz$$

$$=\int\int U\frac{\partial V}{\partial n}dS-\int\int\int U\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}\right)dxdydz \quad (15)$$

$$=\int\int V\frac{\partial U}{\partial n}dS-\int\int\int V\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)dxdydz.$$

The case in which one of the functions, U for example, becomes infinite within the surface S was also considered by Green, and is of prime importance in the present investigation of the theory of globular clusters. To simplify the treatment, suppose U to become infinite at one point P only; then infinitely near this point U may be taken as sensibly equal to $1/r$, where r is the distance from P. Imagine an infinitely small sphere, of radius a, described about P as a center. Equation (15) obviously is applicable to all points exterior to this little sphere. Moreover, since

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{1}{r}=0, \quad (16)$$

it is clear that the triple integral of the right members of (15) may be supposed to extend through the entire enclosed space S, since the part arising from the points within this little sphere is a small quantity of the same order as a^2, and therefore of the second order of small quantities.

Moreover, since near P the function U is sensibly equal to $1/r$, the part of $\int\int U\frac{\partial V}{\partial n}dS$ due to the surface of the sphere is infinitely small of the order of the radius a, which is of the first order of small quantities, and may therefore be neglected. It only remains, then, to consider the part of $\int\int V\frac{\partial U}{\partial n}dS$ due to the spherical surface.

As V is supposed to vary continuously, we may take for it the value V' which is that attained at the point P. Then, since

$$\frac{\partial U}{\partial n}=\frac{\partial U}{\partial r}=\frac{d\left(\frac{1}{r}\right)}{dr}=-\frac{1}{r^2}=-\frac{1}{a^2}, \tag{17}$$

the integral over the sphere $S=4\pi a^2$ will become

$$\int\int V\frac{\partial U}{\partial n}dS=\int\int V\left(-\frac{1}{a^2}\right)d(4\pi a^2)=-4\pi V'. \tag{18}$$

Accordingly, the equation (15) becomes

$$\int\int\int\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\frac{\partial V}{\partial y}+\frac{\partial U}{\partial z}\frac{\partial V}{\partial z}\right)dxdydz$$

$$=\int\int U\frac{\partial V}{\partial n}dS-\int\int\int U\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}\right)dxdydz \tag{19}$$

$$=\int\int V\frac{\partial U}{\partial n}dS-\int\int\int V\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)dxdydz-4\pi V'. \tag{20}$$

In these formulæ, as before, the triple integrals extend through the whole space, and the double integrals over the whole surface. If V had become infinite, instead of U, there would have been the corresponding term $-4\pi U'$ to be added to the right member of (19).

Now in a globular cluster of stars subjected to the mutual gravitation of its components over long ages, many close approaches will eventually develop: and they may depend on the wandering of stars within either the outer shell or the central sphere, or from the shell to the sphere or *vice versa.* Therefore both U and V may become infinite from the appulses of stars under the secular effects of the mutual gravitation of the stars of the cluster. If we denote by $\nabla^2 U$ and $\nabla^2 V$ the Laplacean operation indicated in (16), as applied to the functions U and V, the right members of equations (19) and (20) when modified to include the appulses accumulating in a cluster over long ages, become

$$\begin{aligned}\int\int U\frac{\partial V}{\partial n}dS - \int\int\int U(\nabla^2 V)dxdydz - \sum_{i=1}^{i=i} 4\pi U_i' \\ = \int\int V\frac{\partial U}{\partial n}dS - \int\int\int V(\nabla^2 U)dxdydz - \sum_{i=1}^{i=i} 4\pi V_i'.\end{aligned} \tag{21}$$

In our present problem the triple integrals may be neglected, since $\nabla^2 V$ and $\nabla^2 U$ are each zero, or evanescent, in the small spheres where the appulses occur, and even here are small quantities of the order a^2. Hence by (14) the secular equations become

$$4\pi\Omega = \int\int\int\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial V}{\partial z}\right)dxdydz$$

$$= \int\int V\frac{\partial U}{\partial n}dS - \sum_{i=1}^{i=i} 4\pi V_i' \tag{22}$$

$$= \int\int U\frac{\partial V}{\partial n}dS - \sum_{i=1}^{i=i} 4\pi U_i'. \tag{23}$$

Over very great intervals of time, to be reckoned, as Herschel believed, in "millions of ages," the number of appulses may be taken to be proportional to the time in either the original shell or the original sphere. Consequently instead of the summations in the right members of (22) and (23) we could introduce terms depending directly on the time, and thus write

$$\begin{aligned}4\pi\Omega &= \int\int V\frac{\partial U}{\partial n}dS - 4\pi V'\cdot\alpha(t - t_0) \\ &= \int\int U\frac{\partial V}{\partial n}dS - 4\pi U'\cdot\beta(t - t_0)\end{aligned} \tag{24}$$

where α and β are positive numerical coefficients in the form of undetermined multipliers.

Now it is significant that to the surface integrals negative terms are attached increasing at rates proportional to the time. The second members of (24) cannot therefore be constant, but must decrease with the time. As U and V depend on the coördinates at the initial epoch t_0, and the derivatives $\frac{\partial}{\partial n}$ and $\frac{\partial V}{\partial n}$ depend on the same elements, a progressive decrease in the double integrals, to satisfy the right members of (24), implies that the coördinates of the entire system must so change that the surface S decreases. Thus the globular cluster undergoes a secular compression, owing to the accumulation of appulses, and the shrinkage of the bounding surface.

It is well known that under the operation of universal gravitation the bodies of a system, starting from any initial distribution, tend to fall together, so that the potential energy diminishes. When the number of bodies is very large it becomes impossible for the motions to be simply periodic, like that of a planet or comet moving in a Keplerian ellipse; but although the nature of the non-reëntrant orbits cannot be predicted by any known method, it is possible to say that the potential energy of the system *tends incessantly to a minimum,* while the maximum of the total energy becomes kinetic, and is expended in producing large velocities of the bodies. The left member of (24) therefore incessantly decreases, owing to the exhaustion of the potential energy. This accords with what, on purely mathematical grounds, we found to be the effect of appulses, on the right member of (24). Hence universal gravitation acts as a *clustering power,* and when the figure of a cluster is rendered globular, the dimensions of the system is further diminished under the secular effect of appulses and exchanges of velocities going on within the mass of stars.

In view of the above considerations it is evident that the right member of equations (24) should include independent negative terms to take account of the effect of *general shrinkage,* without regard to appulses due to close approach. Thus the final forms of these equations become:

$$
\begin{aligned}
4\pi\Omega &= \int\int V\frac{\partial U}{\partial n}dS - 4\pi V' \cdot \alpha(t-t_0) - \alpha'(t-t_0) \\
&= \int\int U\frac{\partial V}{\partial n}dS - 4\pi U' \cdot \beta(t-t_0) - \beta'(t-t_0),
\end{aligned}
\tag{25}
$$

where α' and β' are positive numerical coefficients in the form of undetermined multipliers. The first of these negative terms depends on appulses due to close approach, the second on the *general shrinkage* due to the mutual attraction of the stars of the cluster at all distances.

The action of the clustering power upon the figure of a cluster, as Herschel remarked (*Phil. Trans.*, 1789, p. 219), is analogous to that of gravity on the figure of a planet. It might be compared also to the well-known effect of surface tension on the figure of a drop of dew or a drop of mercury, etc. In these last phenomena the *surface is made a minimum,* for a given volume, by the restriction of the elastic layer constituting the outer boundary. A soap bubble is also a good illustration of such *minimal surfaces,* the mathematical theory of which has been placed on a strictly rigorous basis by the researches of the late Professor Weierstrass, one of my most revered teachers at the University of Berlin. In the case of the clusters, however, we have not only a tendency to minimal surfaces, but also for such an internal arrangement of the stars with increase of density towards the center as will reduce the potential energy of the system to a minimum. The theory of the clusters is therefore much more complex than that of simple minimal surfaces, such as we see in drops of dew or soap bubbles, to which the analysis of Weierstrass is applicable.

In the case of the minimal surfaces of the type rigorously treated by Weierstrass, the determination of the minimum is found by the usual condition in the calculus of variations,

$$\delta u = 0, \tag{26}$$

where the function $u = \chi(x, y, z)$ represents the surface.

In the more general problem of clusters, the determination of the minimum potential energy applies to every shell as well as the external surface, and thus for the i concentric shells we have

$$\delta\left(\sum_{i=1}^{i=i} u_i\right) = 0, \tag{27}$$

where $u_i = \chi_i(x, y, z)$ is the equation of any surface.

Moreover, the clusters involve two additional conditions, the first being that in each layer the density σ_i shall depend wholly on the radius, and not at all on the angles (ϕ, θ) usually used in polar coördinates. If any point in any layer be taken as a pole it suffices to regard simply the new polar angle θ; and the required condition is

$$\frac{\delta(\sigma_i)}{\delta\theta} = 0, \tag{28}$$

where $\sigma_i = \psi(r, \theta, \phi)$ is the law of density in any shell, the new angle θ alone being sufficient where there is no fixed pole. The second condition is that the law of density as respects the radius shall be suitable and the same throughout the mass; so that in every part the form of the density function does not vary as respects the radius:

$$\frac{\delta[\psi(r, \theta, \phi)]}{\delta r} = 0. \tag{29}$$

The actual arrangement in any given cluster may not be perfect, but nature always and everywhere works towards the fulfillment of these conditions.

IV. The Observed Law of Density in Globular Clusters.

In the *Monthly Notices* of the Royal Astronomical Society for March, 1911, Mr. H. C. Plummer, of Oxford, has an important paper "On the Problem of Distribution in Globular Star Clusters." For earlier data on the distribution of stars in clusters he refers to a statistical paper by Mr. W. E. Plummer (*Monthly Notices,* June, 1905, Vol. LXV., p. 810), and to the much earlier investigations by Professor E. C. Pickering (*Harvard Annals,* Vol. XXII.) and Professor Solon I. Bailey (*Astronomy and Astrophysics,* Vol. XII., p. 689).

Among the results cited from the researches of Pickering and Bailey are these:

1. The law of distribution is essentially the same for different clusters.

2. The bright stars and faint stars of a cluster obey the same law.

Mr. H. C. Plummer also availed himself of the important researches of Dr. H. von Zeipel on the cluster Messier 3 (*Annales de l'Observatoire de Paris,* Tome XXV.), in which a method was developed for finding the law of distribution of the stars in space, from the observed law of distribution in the projection as we see it. Dr. von Zeipel effected this transformation by means of a theorem due to Abel. He subsequently compared his results for Messier 13 and Omega Centauri with the densities to be expected in a spherical mass of gas in isothermal equilibrium.

In his paper of March, 1911, Mr. Plummer investigates the law of density for the clusters Omega Centauri, 47 Toucani, and the great cluster in Hercules (M. 13). By the use of von Zeipel's method he finds that in these three clusters there is a very good agreement as respects the law of density. In the accompanying table we give the ten points of Plummer's empirical curve of density, based on recent photographs. For the sake of comparison we give also the corresponding points for the laws of density and pressure for a sphere of gas following the monatomic law and in convective equilibrium, as developed in the writer's researches on the "Physical Constitution and Rigidity of the Heavenly Bodies" (*Astron. Nachr.,* Nos. 4053, 4104). The nature of these three laws is best understood from the accompanying illustration, Fig. 1.

1. The cluster density is greater near the boundary, the curve tending to become asymptotic, as there is no definite boundary to the mass of stars.

2. The cluster density also appears to be relatively greater near the center, so that the curve intersects the monatomic curves in the outer parts of the radius but again unites with them at the center, after falling and pursuing a different course between the surface and the center.

3. As the apparent density of the stars in a cluster is consider-

Part of Radius, or Distance from Center	Plummer's Law of Density in Star Clusters	See's Law of Density in Sphere of Monatomic Gas in Convective Equilibrium.	See's Law of Pressure in Sphere of Monatomic Gas in Convective Equilibrinm.
0.0	1.00	1.00	1.00
0.1	0.87	0.97	0.95
0.2	0.63	0.90	0.80
0.3	0.38	0.74	0.60
0.4	0.24	0.58	0.40
0.5	0.145	0.42	0.235
0.6	0.085	0.28	0.118
0.7	0.062	0.161	0.048
0.8	0.035	0.008	0.014
0.9	0.025	0.0025	0.002
1.0	0.020	0.0000	0.000

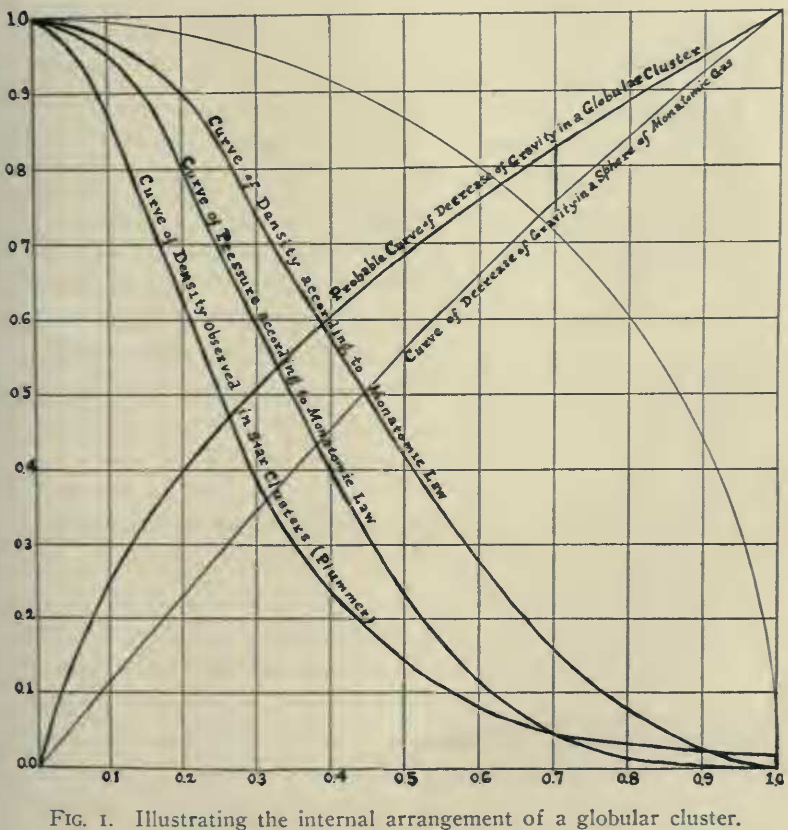

FIG. 1. Illustrating the internal arrangement of a globular cluster.

able, and the images spread somewhat on the plate, it is possible that longer photographic exposures or better plates, on which the images do not spread, would give relatively more stars in the region of the middle of the radius of the cluster, and thus bring the law of density for clusters into essential agreement with the monatomic law of density.

Further photographic observations, with the best modern instruments, alone would decide this question. A final decision can not be made yet, but in order to have the judgment of the best contemporary astronomical photographer on the subject, I have recently referred the question to Professor W. S. Adams, acting director of the solar observatory at Pasadena, who reports as follows:

"I regret that I cannot give answers which would be at all conclusive to your questions regarding the distribution of the fainter stars in star clusters. Up to the present time only a few counts have been made upon our photographs. So far as these go, they do not appear to show any tendency on the part of the fainter stars to predominate around any particular portion of the radius of the cluster, but rather for the distribution to be tolerably uniform. The problem is made difficult by the fact that the central part of our photographs is almost always burned out, so that counting is impossible for some distance along the radius. We have begun, however, to take series of photographs of clusters, giving exposure times with a ratio of 1 to 2.5. These should help greatly in providing an answer to your questions."

On the whole the indications are that the capturing process of drawing in stars from without is still going on. This would account for the small density near the outside of the cluster, and also the great central density, the latter being an accumulative effect of the various shells in the course of millions of ages.

V. The Potential due to a Mass of Globular Figure Assumed under the Action of Central Powers.

In my "Researches," Vol. II., 1910, I have outlined the process by which the nebulæ form by the aggregation of dust from a distance; and shown that the collecting streams may often take the spiral form, and in this early stage are not of symmetrical figure. The general integrals in Section II. are required to express the attraction of these unsymmetrical masses. But in true sidereal systems as old and fully developed as the globular clusters are known to be, a state of very perfect symmetry has been attained through the oscillations of the entire mass, and the mutual adjustments of the parts of the system, and by the rounding up of the orbits under the secular action of the resisting medium, as implied in Plato's remark that the Deity always geometrizes—ὁ θέος ἀεὶ γεωμέτρει. On this latter process I have dwelt at some length in an address on "The Foundations of Cosmogony," delivered to the St. Louis Academy of Sciences, May 1, 1911, and printed in the *Memorie delle Societa degli Spettroscopisti italiani,* Rome, Vol. XL., 1911; and in another address entitled "The Evolution of the Starry Heavens," delivered to the California Academy of Sciences, Aug. 7, 1911, and printed in *Popular Astronomy* for November and December, 1911.

Herschel's theory of the spherical figures of clusters (*Phil. Trans.,* 1789, p. 217), conceived as made up of a series of concentric shells of uniform density, but with increasing accumulation towards their centers, is confirmed by modern photographs of various clusters as shown in the accompanying plates from my "Researches," Vol. II. The attraction of a mass of this kind thus becomes similar to that of a sphere made up of concentric homogeneous layers, but with the density increasing towards the center. The integration for the central attraction in these perfectly symmetrical figures thus need not involve θ or ϕ, but only the radius r.

If σ_0 be the central density of the cluster, and σ the density at any point whose distance from the origin of the coördinates at the center is x, a shell of density σ and thickness dx will have the mass

$$dm = 4\pi\sigma x^2 dx. \tag{30}$$

And the sphere enclosed by this shell will have the mass

$$m = 4\pi \int_0^x \sigma x^2 dx = \tfrac{4}{3}\pi\sigma_1 x^3, \tag{31}$$

where σ_1 is the average density of the enclosed layers included between $x = 0$ and $x = x$. Thus we have

$$\sigma_1 = \frac{3m}{4\pi x^3} = \frac{3\sigma_0}{x^3} \int_0^x \left(\frac{\sigma}{\sigma_0}\right) x^2 dx. \tag{32}$$

At the surface of the cluster the gravity of the entire mass will become

$$G = \frac{M}{x'^2}, \tag{33}$$

where M is the mass of all the stars and x' the exterior radius of the cluster. If G' be the value of the force of gravity of the cluster at any point below the surface, at a distance x from the center, we shall have

$$G' = \frac{4\pi\sigma_0}{x^2}\int_0^x \left(\frac{\sigma}{\sigma_0}\right) x^2 dx. \tag{34}$$

The outer shell of the cluster is here neglected as exerting no attraction on a point within, as was long ago established by Newton for homogeneous solid bodies (cf. "Principia," Lib. I., Prop. XCI., prob. XLV., Cor. 3).

To find the ratio of G' to G so as to give the law of central force within the cluster, we have the relation

$$\frac{G'}{G} = \frac{\frac{4\pi\sigma_0}{x^2}\int_0^x \left(\frac{\sigma}{\sigma_0}\right) x^2 dx}{\frac{4}{3}\pi\sigma_1' \cdot \frac{x'^3}{x'^2}} = \frac{3\sigma_0 \int_0^x \left(\frac{\sigma}{\sigma_0}\right) x^2 dx}{x' x^2 \frac{3\sigma_0}{x'^3}\int_0^{x'} \left(\frac{\sigma}{\sigma_0}\right) x^2 dx} = \frac{x'^2 \int_0^x \left(\frac{\sigma}{\sigma_0}\right) x^2 dx}{x^2 \int_0^{x'} \left(\frac{\sigma}{\sigma_0}\right) x^2 dx}. \tag{35}$$

The evaluation of this ratio depends on the integrals between the assigned limits, one corresponding to the entire sphere of radius x', and one to the part of the sphere included within the radius x. Thus the integrals depend on the law of density in the cluster. We have already seen from the researches of Dr. H. von Zeipel, and Mr. H. C. Plummer that the accumulation of density towards the center appears to slightly exceed that of a sphere of monatomic gas in convective equilibrium and fulfilling adiabatic conditions (*A. N.*, 4053, and *A. N.*, 4104).

Although the monatomic law may not hold strictly true in clusters, yet it seems worth while to examine the results which will follow from this law. In *A. N.*, 4053, p. 327, it is shown that

$$\frac{\sigma}{\sigma_0}=\frac{1}{x^2}\frac{d\mu}{dx}, \tag{36}$$

where the expression for μ, with the correction noted in *A. N.*, 4104, p. 386, is

$$\begin{aligned}\mu=&\frac{x^3}{3}-\frac{x^5}{20}+\frac{x^7}{240}-\frac{x^9}{3888}+\frac{19x^{11}}{1425600}-\frac{2719x^{13}}{4447872000}\\&+\frac{20621x^{15}}{800616960000}-\frac{193328x^{17}}{190546836480000}\\&+\frac{39667364x^{19}}{1042672289218560000}-\frac{8078124341x^{21}}{591195187986923520000}+\cdots.\end{aligned} \tag{37}$$

Now if we substitute the value of σ/σ_0 from (36) in the integrals of (35), they are reduced to two series which may be called μ and μ', the latter having the same form as (37) but the limit x' instead of x. Accordingly (35) becomes

$$\begin{aligned}\frac{G'}{G}=\frac{x'^2\mu}{x^2\mu'}&=\frac{x'^2\left[\frac{x^3}{3}-\frac{x^5}{20}+\frac{x^7}{240}-\frac{x^9}{3888}+\cdots\right]}{x^2\left[\frac{x'^3}{3}-\frac{x'^5}{20}+\frac{x'^7}{240}-\frac{x'^9}{3888}+\cdots\right]}\\&=\frac{x\left\{\frac{1}{3}-\frac{x^2}{20}+\frac{x^4}{240}-\frac{x^6}{3888}+\cdots\right\}}{x'\left\{\frac{1}{3}-\frac{x'^2}{20}+\frac{x'^4}{240}-\frac{x'^6}{3888}+\cdots\right\}}.\end{aligned} \tag{38}$$

As the coefficients of the series μ and μ' are the same, we may calculate from the equation (37) or (38) the value of the ratio at suitable intervals throughout the sphere, and ascertain rigorously the law of the variation. The results of my calculations are given in the following table and illustrated by the corresponding curve in Fig. 1.

TABLE SHOWING DECREASE OF CENTRAL GRAVITY IN A SPHERE OF MONATOMIC GAS IN CONVECTIVE EQUILIBRIUM. THE DECREASE OF CENTRAL GRAVITY IN A GLOBULAR CLUSTER IS SLIGHTLY LESS RAPID, OWING TO GREATER ACCUMULATION OF DENSITY TOWARDS THE CENTER.

Distance from Center. $=\frac{x}{x'}$,	Ratio of Internal Gravity to Surface Gravity. $=\frac{G'}{G}$
1.0	1.00000
0.9	0.92563
0.8	0.84378
0.7	0.75495
0.6	0.65975
0.5	0.55928
0.4	0.45317
0.3	0.34769
0.2	0.23069
0.1	0.11587
0.0	0.00000

VI. DYNAMICAL STATE OF A GLOBULAR CLUSTER.

In works on the theory of potential and attraction the following theorems are demonstrated and well known:

1. That a *sphere* either homogeneous or made up of concentric spherical shells attracts an external point as if collected at its center (Newton's "Principia," Lib. I., Prop. LXXVI., Theorem XXXVI.).

2. That *homogeneous spherical shells* attract external points as if collected at their centers of figure, and exert no attraction on points within ("Principia," Lib. I., Prop. LXX., Theorem XXX.). Also a point within the sphere is attracted by a force proportional to the distance from the center ("Principia," Lib. I., Prop. LXXIII., Theorem XXXIII.); and the same theorem holds for the spheroid made up of concentric spheroidal shells ("Principia," Lib. I., Prop. XCI., Prob. XLV.).

3. That *ellipsoidal homœoids,* or ellipsoidal shells of any thickness made up of homogeneous layers, bounded by two ellipsoidal surfaces, concentric, similar and similarly placed, likewise exert no attraction on points within, as is shown by Newton in the "Principia" (Lib. I., Prop. XCI., Prob. XLV., Cor. 3) for the case of the spheroid, which corresponds to the figure and internal arrangement of density in such bodies as the planets, sun, and stars.

To illustrate the simple case of a *homogeneous sphere,* we remark that it attracts a point at its surface with a force

$$f = \frac{4}{3}\frac{\pi\sigma r^3}{r^2} = \tfrac{4}{3}\pi\sigma r = Cr, \tag{39}$$

where σ is the density, and r the radius. This equation shows that all points within the sphere are attracted to the center by forces proportional to the radii of the shells on which they are situated, since the external shells exert no attraction on points within.

Let the solid sphere be set rotating steadily about an axis; then as the central forces at the various points are proportional to the radii described by the points, there will be no tendency arising from the central attraction for any shell to be displaced with respect to the shells within or without, once the condition of equilibrium is attained, but the central accelerations will everywhere tend to secure steady motion without relative displacement of the parts of the sphere. The same is true of the centrifugal force, after the adjustment to a suitable figure of equilibrium; for the centrifugal force is equal to v^2/r, v being the velocity of the particle and r the radius it describes; for this gives

$$f = \frac{v^2}{r} = \frac{(2\pi r)^2}{t^2 r} = \frac{4\pi r}{t^2}; \tag{40}$$

and as it is common for all particles the force has the same form here as in equation (39).

What is here proved for the simple case of the homogeneous sphere, will obviously hold also for a sphere made up of concentric spherical shells of uniform density; for the theorem will hold for all the points within. And similarly for *ellipsoidal homœoids,* or spheroids such as the planets, sun and stars. If any of these masses have attained uniform movement as of rotation, there is no tendency to produce a relative displacement of the parts.

Now the simple equation (39) shows that a similar theorem holds for the internal dynamics of a globular cluster, the component stars of which have attained a state of equilibrium following a definite law of density depending only on the radius. But before treating of this at length, we shall recall a suggestive investigation of Sir William Herschel printed in the *Philosophical Transactions* for 1802 (pp. 477–502) under the title "Catalogue of 500 New Nebulæ, Nebulous Stars, Planetary Nebulæ, and Clusters of Stars; with Remarks on the Construction of the Heavens."

VII. Herschel's Theorem on the Motion of Multiple Stars, 1802.

In the important paper just cited Herschel first discusses "Binary Sidereal Systems or Double Stars," and then proceeds to Section "III. Of more complicated sidereal systems, or treble, quadruple, quintuple and multiple stars," where he reasons as follows:

"In all cases where stars are supposed to move round an empty center, in equal periodical time, it may be proved that an imaginary attractive force may be supposed to be lodged in that center, which increases in a direct ratio of the distances. For since, in different circles, by the law of centripetal forces, the squares of the periodical times are as the radii divided by the central attractive forces, it follows, that when these periodical times are equal, the forces will be as the radii. Hence we conclude, that in any system of bodies, where the attractive forces of all the rest upon any one of them, when reduced to a direction as coming from the empty center, can be shown to be in a direct ratio of the distance of that body from the center, the system may revolve together without perturbation, and remain permanently connected without a central body."

This reasoning is best understood by means of simple formulæ: Let f_1 and f_2 be two centrifugal forces, which in revolving systems are always equal to the centripetal forces, and V_1 and V_2 the corresponding velocities of the bodies, and r_1 and r_2 the radii of the circles in which they are supposed to revolve. Then, by the elementary principles of mechanics, we have

$$f_1 = \frac{v_1^2}{r_1}; \; f_2 = \frac{v_2^2}{r_2}; \text{ whence } f_1 = \frac{(2\pi r_1)^2}{t_1^2 r_1}; \; f_2 = \frac{(2\pi r_2)^2}{t_2^2 r_2}.$$

This gives

$$t_1^2 = \frac{4\pi^2 r_1}{f_1}; \; t_2^2 = \frac{4\pi^2 r_2}{f_2}. \tag{41}$$

Now in orbital revolution the centripetal and centrifugal motions are always exactly equal, and hence if $t_1 = t_2$, we have

$$\frac{t_1^2}{4\pi^2} = \frac{r_1}{f_1}; \quad \frac{t_2^2}{4\pi^2} = \frac{r_2}{f_2}, \quad \text{or} \quad \frac{r_2}{f_2} = \frac{r_1}{f_1},$$

whence

$$\frac{f_1}{f_2} = \frac{r_1}{r_2}, \tag{42}$$

as concluded by Herschel in the *Philosophical Transactions,* for 1802, p. 487.

To establish clearly such actual cases of motion, with the attractive force in the direct ratio of the distance from the empty center, where he says the system may revolve together without perturbation, and remain permanently convected without a central body, Herschel proceeds to deal first with two equal double stars revolving in circles about the common center of gravity of the system. He next generalizes the procedure by taking two unequal masses, then treats also the cases of motion in elliptic orbits, and finally considers certain types of triple and multiple stars, to which similar reasoning will apply. This paper of Herschel is quite remarkable, and deserving of more attention than it has received.

VIII. Theorem on the Revolutions of Stars in Clusters.

It is now obvious that the clusters which have attained a definite law of density depending wholly on the radius will conform to Herschel's Theorem of motion about empty centers, which is also the law for the central motion of particles of a rotating solid. If we imagine a heterogeneous sphere made up of concentric homogeneous layers, but with the density of the layers increasing towards the center, and take the radii of the layers to be $r_1, r_2, r_3, \ldots r_i$, and denote by $\sigma_1, \sigma_2, \sigma_3, \ldots \sigma_i$ the average density of the sphere up to the ith layer inclusive; then the attraction on points in these several layers will be $A_1, A_2, A_3, \ldots A_i$, as follows:

$$A_1 = \frac{4}{3}\frac{\pi\sigma_1 r_1^3}{r_1^2} = C_1 r_1; \; A_2 = C_2 r_2; \; A_3 = C_3 r_3; \; \cdots; \; A_i = C_i r_i. \tag{43}$$

Thus the constant will vary from layer to layer in a heterogeneous sphere made up of concentric homogeneous shells, but the attraction at every point, including the external surface, is proportional to the radius of the shell in question.

Now just as a sphere, either homogeneous or made up of concentric layers of uniform density, attracts all internal points, including those at the external surface, with a force proportional to the radius of the shell on which it is situated; so also will a cluster which is condensed towards the center according to any law of density depending wholly on the radius, attract all internal points, including those in the external surface, according to the same law of direct proportionality to the distance from the center.

When the attractive force varies directly as the distance from the center, the particle so attracted describes an ellipse as was first proved by Newton in the "Principia" (Lib. I., Prop. X., Prob. V.). This case of attraction depending directly on the first power of the distance is also discussed by the analytical method in Vol. II. of my "Researches," 1910, pp. 25–27, where it is shown that the time of revolution is quite independent of the dimensions of the ellipse, but depends wholly on the intensity of the central force.

For motion in a plane the coördinates of the particle are shown to be defined by the equations:

$$x = \frac{V\cos\psi}{\sqrt{\mu}}\sin\sqrt{\mu}t + a\cos\sqrt{\mu}t, \quad y = \frac{V\sin\psi}{\sqrt{\mu}}\sin\sqrt{\mu}t. \quad (44)$$

As the values of the coördinates are the same at the time t and $t + \frac{2\pi}{\sqrt{\mu}}$, it is evident that the time of revolution is $\frac{2\pi}{\sqrt{\mu}}$, or inversely as the square root of μ, where μ is the mass, and exerts the corresponding unit of force at unit distance.

In a cluster with stars arranged according to a law of density depending wholly on the radius, the value of μ or the force will depend wholly on the radius also, as shown in equation (34). And thus the time of revolution will be independent of the dimensions of the ellipse. Assuming that there is but little relative displacement of the bodies of the clusters, a star situated, therefore, in an outer shell will revolve about the common center of gravity in exactly the same time as one situated near the center; for the remoter stars revolve under a greater attractive force, while the nearer ones revolve under feebler forces, and all would therefore have a common period. The movement at the end of the period would restore the cluster to its original state, the individual bodies being exactly where they started from at the initial epoch. This is one of the most remarkable results of the dynamics of a system of n-bodies arranged in concentric shells of uniform density, depending wholly on the radius, as in our typical globular clusters, which are made up of stars of equal brightness and apparently of equal mass.

If therefore the cluster were once established with such relations among the stars that their orbits do not intersect, and the sphere of powerful attraction for each star is small compared to the spaces between the neighboring stars, the wonderful system thus arranged might oscillate in stability for millions of ages. These conditions evidently are quite fully realized in the globular clusters, as will more clearly appear from the following considerations on their mode of formation.

IX. The Symmetrical Growth of a Cluster due to a Process of Internal Compensation.

In the second volume of my "Researches," 1910, it is shown by a line of argument based on the principle of continuity, similar to that used by Herschel in the *Phil. Trans.*, 1811, p. 284, that the nebulæ are formed by the gathering together of dust expelled from the stars under the action of repulsive forces. As this dust gathers towards a center so as to form a nebula or cluster, of more or less symmetrical figure, it takes a long time for the new system to acquire an arrangement by which the density increases from the surface to the center. In the course of ages, however, the central mass increases or the central group of masses accumulate, by accretion of dust to the individual bodies, or by the capture and redistribution of interpenetrating bodies. The result, on the one hand, is that all orbits will be decreased in size and the system will contract its dimensions; and, on the other hand, this waste matter will tend to accumulate in regions of stability, and there build up the smaller into larger bodies. Thus the individual stars being supplied from such varied sources the cluster will necessarily acquire increasing symmetry, and orderly arrangement, like those actually observed.

This natural tendency to order and stability will be greatly augmented by mutual compensation among the stars of the cluster. As the stars are both gaining and losing matter incessantly, under the mutual interaction of attractive and repulsive forces, it is evident that those which gain too rapidly, will also begin at once to lose at an abnormal rate, owing to the augmented action of the repulsive forces; and the dust expelled from them will go directly or indirectly very largely to the other members of the cluster, and thus operate to restore the equilibrium of the whole group. Moreover, if any serious collision occurs, by which one star acquires predominant size, it will at the same time acquire such abnormal energy of radiation, that the balance of power will tend to become gradually restored under the action of the repulsive forces at work.

From these known causes one would expect a cluster therefore to be a mutually compensating system, producing and building up new bodies in vacant regions, where the conditions are stable, and redistributing undue accumulations of mass by the natural balance established between attractive and repulsive forces, as all the stars gain matter from surrounding space and again expel it after a certain repulsive vigor has been attained. The eventual accumulation of so many stars in a comparatively small space largely operates to retain the dust expelled from them in that region; it thus goes to other members of the group, rather than to the rest of the remote stars of the universe, so that in the course of vast time—millions of ages in Herschel's expressive phrase—the cluster accumulates to such grandeur and order as we see in such noble globular clusters as that in Hercules, 47 Toucani, and Omega Centauri.

It is worthy of note that this simple theory, based on known and established laws, explains not only the origin and growth of these wonderful masses of stars, under conditions of stability; but also the nearly perfect equality of the individual stars which has always been so bewildering to astronomers.

X. How a Star Entering a Cluster has its Oscillations Damped and is Finally Captured.

If we recall the familiar equations for an oscillation, as treated in works on physics,

$$\eta = ae^{-kt} \cos(nt + \alpha), \text{ or } \eta = ae^{-kt} \sin(nt + \beta), \qquad (45)$$

where a is the original amplitude of the harmonic oscillation, so that ae^{-kt} becomes a coefficient decreasing as t increases, $n = 2\pi/T$, T being the period; we see that as the time t increases the ordinate η will decrease, though the period T remains constant. The equation (45) thus represents a *damped vibration,* such as constantly arises where resistance is encountered by vibratory motion. Under these circumstances the harmonic curve rapidly loses amplitude and is of the form:

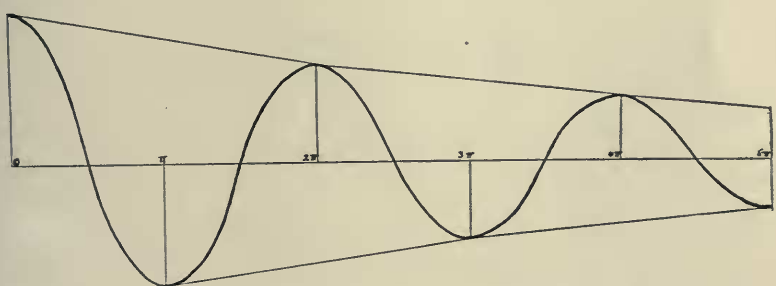

Fig. 2. Illustrating damped vibrations.

The process of damping here brought to light for oscillating particles describing simple harmonic motion has its analogies in the movements of stars in a cluster; for here too the period of the movement, as we have seen in VIII., is essentially constant, but the amplitude of the oscillation is reduced till it becomes adapted to that of the rest of the system. This is a part of the capture process, because it tends to reduce all the abnormal movements to one dead level.

Let us now examine the dynamical process by which stars tend to become entrapped in the central region of a cluster. If we consider the potential of a spherical shell of stars obeying any law of density, and having a thickness R-r, it is evident from equation (7) that

$$V' = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_r^R \sigma r dr, \tag{46}$$

and the forces along the coördinate axes will be

$$\begin{aligned} X' &= \int_0^{2\pi} \cos\phi d\phi \int_0^{\pi} \sin^2\theta d\theta \int_r^R \sigma dr, \\ Y' &= \int_0^{2\pi} \sin\phi d\phi \int_0^{\pi} \sin^2\theta d\theta \int_r^R \sigma dr, \\ Z' &= \int_0^{2\pi} d\phi \int_0^{\pi} \cos\theta \sin\theta d\theta \int_r^R \sigma dr. \end{aligned} \tag{47}$$

Now every globular cluster may be regarded as made up of a series of such shells; so that the total forces become

$$\begin{aligned} X &= \sum_{i=1}^{i=i} X_i = \sum_{i=1}^{i=i} \int_0^{2\pi} \cos\phi d\phi \int_0^{\pi} \sin^2\theta d\theta \int_r^R \sigma dr, \\ Y &= \sum_{i=1}^{i=i} Y_i = \sum_{i=1}^{i=i} \int_0^{2\pi} \sin\phi d\phi \int_0^{\pi} \sin^2\theta d\theta \int_r^R \sigma dr, \\ Z &= \sum_{i=1}^{i=i} \quad = \sum_{i=1}^{i=i} \int_0^{2\pi} d\phi \int_0^{\pi} \cos\theta \sin\theta d\theta \int_r^R \sigma dr. \end{aligned} \tag{48}$$

These expressions are so complex, that we are obliged to restrict our consideration to the action of a single shell. Accordingly, we shall suppose the single shell filled with stars to a considerable density, and the distribution uniform. An external star coming in from the distance, if otherwise undisturbed, will revolve in a Keplerian ellipse having its focus in the center of the shell. The mass acting as if collected at the center is $4\pi\sigma r^2(R-r)$, where the thickness R-r is not too large; and the velocity acquired at the outer border of the shell is

$$V^2 = k^2[4\pi\sigma r^2(R-r)]\left[\frac{2}{R} - \frac{1}{a}\right], \tag{49}$$

where a is the semi-axis major of the Keplerian ellipse.

As soon as a part of the shell of thickness dr has been traversed, however, the stars included in the space $4\pi r^2 dr$ will cease to exert attraction on the moving star; and the further it enters the shell the less central attraction will be exerted from the original focus. As the star quits the shell and enters the hollow space within there will be no central attraction to cause it to describe a Keplerian ellipse. Thus as the radius vector decreases from R to r, the path ceases to be the arc of an ellipse, and becomes a straight line. The body thus moves uniformly across the hollow of the shell, and enters again on the opposite side, with the same velocity it had on quitting the shell. The central attraction of the shell begins to be felt as soon as a layer of thickness dr is traversed, for the space $4\pi r^2 dr$ has a mass $4\pi\sigma r^2 dr$, and it attracts as if collected at the center of the shell. This force grows till the star emerges from the shell on the outside, when it is equal to that operating at the moment the star first entered the shell. Consequently it will depart from the shell on a Keplerian ellipse exactly similar to that on which it first came in; and the *total external orbit* will consist of two exactly similar and similarly situated parts of ellipses, joined by straight lines in the hollow of the shell, and within each layer of the shell gradually passing from the arc of an ellipse to a straight line. This path is illustrated by the accompanying figure.

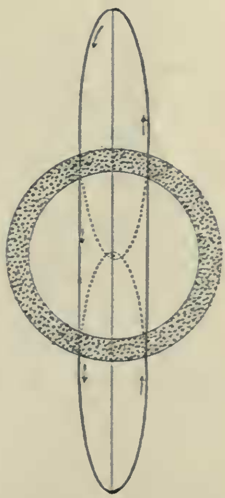

FIG. 3. Illustrating the capture of an oscillating star by the action of a spherical shell uniformly filled with stars as in a globular cluster.

The orbit here described supposes that no local perturbations have occurred during the complete revolution. Let us now consider the *average effect* of such perturbations as will occur. These may be best understood by analogy with the average effect of Jupiter on comets crossing his orbit. It is will known that many comets originally traveling in orbits almost parabolic have been thrown within Jupiter's orbit, till quite a large family has been acquired with short periods, and aphelia near Jupiter's path; and those which still overlap his orbit are being gradually worked into the more stable region within the orbit of that giant planet. In the same way the asteroids have been thrown within Jupiter's orbit, as H. A. Newton justly remarked in 1894 (cf. my "Researches," Vol. II., 1910, p. 699), by a process which Professor E. W. Brown has more fully investigated in the *Monthly Notices* for March, 1911. Professor H. A. Newton's researches and those of Callandreau and Tisserand on the capture of comets are well known, and need not be described here.

Now if for Jupiter we substitute the action of the shell of the cluster, it may be thought that Jupiter is a very large mass, while the comets are very small; whereas the stars in the shell of the clusters are not supposed to be so much larger than the star falling in. This is very true, but as the shell contains many stars in mutual adjustment to an average state of stability, the oscillating star in the course of ages will be disturbed by the many stars, and the cumulative effects will be added together, just as the actions on comets are by the massive planet Jupiter. The mass of the shell greatly exceeds that of the single oscillating star, and even if some of the individual stars in the shell are considerably disturbed, yet the disturbance in successive revolutions will not effect the same stars, owing to movements within the shell of the cluster; and thus in the long run the only possible effect of the action of the many upon the one visiting star will be to dampen its energy of oscillation, till it too will have its path reduced and take its place in the shell with the original group. Thus the visitor from without is entrapped and its movements dragged down to the dead level of the rest of the stars in the shell.

This is a general explanation of the capture process established by the more rigorous method of integration depending on Green's theorem, when some of the terms become infinite. It seemed desirable to examine the matter from both points of view.

To be sure this transformation may take many millions of years, but the average effect of the action of the shell in the long run is certain. As the stars in the shell are comparatively quiescent, the only possible average effect of their action on the visitor will be to exert a drag on its motion. Some of the quiescent stars may be slightly disturbed by the passing body, but as the effect of one appulse is likely to be comparatively small, the stars in the shell will readjust the relations among themselves easily, while the visitor will suffer a considerable retardation of its oscillation. And after many appulses the visitor will have its motion restricted to the shell like the motion of the multitude of stars composing it.

This explains in a simple manner the capture process by which clusters are built up, and given such accumulation of density towards their centers. For the clusters are made up of a series of shells, and if the effect of one shell is of this type, the effect of all the shells will be an integration of these damping effects. It is no wonder therefore that all the clusters show such pronounced accumulation of density towards their centers. It is the inevitable outcome of this capturing of foreign bodies in the course of immeasurable time.

In section III. we have admitted the possibility that defects in our photographs will account for the central density in clusters exceeding that of the atoms in a globe of monatomic gas in convective equilibrium; but in view of this capture process, it seems much more likely that the stars are accumulating in these centers beyond the normal density for a mass of monatomic gas. Thus have the clusters been built up to such extraordinary accumulation that they justly excited the wonder of Sir William Herschel.

XI. The Globular Clusters can be Explained Only by the Capture Theory.

The figures of the clusters, nebulæ and other sidereal systems impressed Herschel with the view that there is a clustering power in nature, everywhere gathering the stars into globular swarms, and moulding the nebulosity into figures of greater and greater symmetry (*Phil. Trans.*, 1789, pp. 217–219). This is the earliest outline of the modern capture theory as applied to clusters and nebulæ of symmetrical figure. It is evident that this process gives a good explanation of the origin of the clusters, and that they can be explained in no other way.

It is obvious that masses of such vast extent and perfectly round figure and symmetrical arrangement of internal density, could not possibly have arisen by any of the theories of collision formerly held but now abandoned. For collisions could not disperse the stars to such great distances over spaces measured by many thousands of light years, nor could they give rise to the observed symmetrical arrangement of the parts. Moreover, clusters embracing thousands of stars, if due to collision, would imply two equally immense masses in collision; and there would be so few of these large masses in the universe, that it is inconceivable that they would ever come into collision. The whole collision doctrine is manifestly inconsistent with the symmetry and order found in the clusters, which can therefore be explained only by the capture theory, based on the expulsion of dust from the stars, and its collection from all directions into masses of impressive symmetry.

This theory not only gives a perfectly satisfactory account of the phenomena of the clusters, which are wonderful in the extreme, and show steady and uniform processes working slowly over immeasurable ages; but also establishes the theory itself by the way the most intricate and diverse phenomena are woven into a continuous whole.

The first rule of philosophy laid down by Newton in the "Principia" is that: "*We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances.*" He explains this by adding that "philosophers say nature does nothing in vain, and more is in vain when less will serve." The next rule is that we are to ascribe the same natural effects to the same causes.

If therefore the capture theory alone will explain the clusters, where the scale of the operations is immense, and the symmetry so perfect that other causes are easily excluded; and on the other hand it will equally account for all other known phenomena of the sidereal universe, it follows from Newton's rules for philosophizing that this cause alone can be regarded as established. The definite proof of the capture theory for the formation of clusters and nebulæ thus renders its operation general throughout the sidereal universe. Everywhere the large masses drift towards the most powerful neighboring center of attraction, while fine dust is expelled from the stars to produce nebulæ in the vacant regions of the heavens; and this concentration of the large masses under gravity, and the redistribution of the fine dust by the action of repulsive forces is the great law of nature which preserves the order of the starry heavens.

XII. The Mutual Interaction of Attractive and Repulsive Forces Confirmed by a Delicate Criterion Based on the Exact Equality of Thousands of Associated Stars in a Cluster.

Elementary considerations on the principles of probability will show that the chances of even two associated stars being of equal brightness is slight; it is still smaller for three, four and higher multiples, and when the number becomes large the probability of the chance association of such equal stars totally disappears. Accordingly, it is not by accident that thousands of stars observed in a cluster, with perfectly symmetrical accumulation towards the center of the associated stars, all known to be at nearly the same distance from us, are as exactly equal in every respect as the finest coins turned out of a mint. There must be in nature a reliable process for the manufacture of these nearly equal stars, which is described above for the first time.

To prove this more conclusively we may compare clusters with double and multiple stars, which are systems of lower order. In binaries the components often are very unequal in brightness, and also in mass. The same principle, as is well known, holds for triple and quadruple stars. Now in these double and multiple star systems the ratio of the mass of the components depends on the chance division of the original nebulosity gathered from the heavens, not from the associated stars themselves; but in the clusters the *principle of redistribution* becomes largely predominant, owing to the great number of radiating centers in close association. It is not surprising, therefore, that the lower orders of stellar systems should include: *first, single stars,* with planetary systems, amounting to about four fifths of all the stars; *second, binary stars,* with unequal components; *third, multiple stars,* also with components very unequal. This inequality of the associated stars is to be expected in all sidereal systems made up of a small number of bodies, but the visual double stars, as the brighter and more easily recognized systems, appear to have components more nearly equal than the much greater number of systems,* which remain invisible at the distance of the fixed stars.

The association of *thousands of equal stars in a cluster* must therefore depend on something besides a chance distribution, or partition of the primordial nebulosity. For although the clusters are very far away, and the double stars in a cluster would thus appear single from the perspective effect of distance alone, yet the distance would not prevent fainter single stars from appearing on the background of the cluster if they were present. Perrine points out in Lick Observatory Bulletin No. 155 that in cluster there is rarely a difference of more than two magnitudes among the stars composing it. This difference probably depends on difference in the spectral types, rather than on difference in mass. The conclusion that the great equality in luster depends on the essential equality in the redistribution of dust within the system therefore seems unavoidable, as a necessary result of known laws of nature actually proved to be in operation. If therefore this argument regarding the origin of clusters, based on the equality of the stars, is admissible, the explanation may as confidently be depended on as the law of gravitation itself. For the testimony of the sidereal universe to its truth seems to be absolutely overwhelming. There are in all over one hundred globular clusters, and they include millions of stars; so that the observed order of nature obviously rests on a fundamental cause.

Accordingly, if we admit the truth of this theory of clusters, which now seems to be well established, through the evidence presented by hundreds of globular clusters, and by the analogous evidence offered by thousands of nebulæ, we have at the same time an equally satisfactory proof of the universality of the operation of repulsive forces in nature. With his usual penetration Herschel saw in the accumulation of density and brightness towards the centers of these masses an incontestible proof of the existence of a clustering power operating throughout the sidereal universe.

Now by *exactly reversing his argument* we have an equally valid proof of the operation of repulsive forces, to give the original distri-

* Resembling planetary systems.

bution of dust, out of which the clusters and nebulæ are finally built. Moreover, as already remarked, this general argument, drawn from the sidereal universe as a whole, is minutely verified in the construction of clusters, by the exact equality of thousands of closely associated stars, which thus supply a criterion of unrivaled rigor. This cluster criterion authorizes the conclusion that the theory may now be removed from the category of speculation and entered in the list of established facts relating to the physical universe. The most obvious indications of nature are plain enough; and in interpreting them all we need to do is to follow the theory of probability, which, as Laplace has remarked, is nothing but common sense reduced to calculation. This theory tells us that there is a deep underlying cause for the perfect equality of the associated stars in clusters, which can be nothing else than the mutual interaction of attractive and repulsive forces in these island universes.

XIII. The Real Dimensions of the Clusters and the Average Distances of the Stars Apart.

The question of the distances of the clusters is one which at present cannot be fully answered, owing to the lack of certain observational data; but it is well known that nearly all these masses of stars are very remote. To be sure such an outspread swarm as Coma Berenices, really is a cluster so near us as not to be suspected of belonging to the same type as the better defined Pleiades, Praesepe and Omega Centauri. But leaving out of account a few exceptions of this class we may say that the globular clusters in general, like the nebulæ considered by Dr. Max Wolf in *A. N.*, 4549, are thousands of light-years in diameter. This is proved by the comparative faintness of the component stars, and the large angular magnitude of the clusters as seen in the sky.

Accordingly, even when there are thousands of stars in a very compressed cluster, they are not really close together, but separated by great intervals, of the order of a light-year. Thus the components in a dense cluster probably are somewhat closer together than our sun is to Alpha Centauri; and yet the intervals can hardly be less than a ten thousand fold radius of the earth's orbit, the light-year being 63,275 times that distance. In fact the average distances are likely to be several light-years, and thus of the order 100,000 radii of the earth's orbit.

This great distance of the stars apart, even in the densest cluster, will enable us to realize the well-known fact that our sun is in a solar cluster, which includes Sirius, the stars of Ursa Major, and many other bright objects. It also enables us to appreciate why the motion in the clusters necessarily is slow, owing to the great intervening spaces and the feebleness of the disturbing forces acting on the individual stars. And at the same time we easily see why such a system, under the mutual gravitation of its parts, might survive for infinite ages, without sensible decay of its order or stability. Newcomb therefore was right when he remarked that there might be planets revolving about the stars in a cluster (article "Stars," Encyclopedia Americana); for we ourselves live on a planet attached to a star of the solar cluster, and the other clusters of the sidereal universe are not very different from that including our sun.

Sir William Herschel was of the opinion (*Phil. Trans.*, 1789, p. 225) that the clusters which are most compressed are drawing on towards a period of dissolution. In an earlier paper of 1785 Herschel suggested that the clusters are the laboratories of the universe where the most salutary remedies for the decay of the whole are prepared (*Phil. Trans.*, 1785, p. 217).

In my "Researches," Vol. II., 1910, I have independently pointed out that the condensation of very compressed clusters into one mass is the only logical explanation of such immense stars as Canopus and Arcturus. For it appears that with the advance of age the state of compression slowly increases, and when it has become extreme, and all the single bodies are drawn very near the center, it is quite likely that the cluster by conflagration may become the furnace of a *laboratory* of the universe for repairing through repulsive forces the ravages wrought by universal gravitation in the course of millions of ages.

If this be true someone may ask why we do not find some cluster in the stage of conflagration? But if we recall that only a little over one hundred globular clusters are known, with their internal spaces still large, and remember also the vast interval of time required to produce the *invisible state of close compression,* it will become evident that the chances of our living at the epoch of a cluster conflagration totally disappears, and the most we can hope to recognize is the resulting giant star such as Canopus.

XIV. Proof that Matter Actually is Lost from our Sun Supplied by the Very Straight Tails of Comets Developed in Close Perihelion Passage.

In view of the recent development of the doctrine of repulsive forces in nature, it becomes important to have readily at hand specific illustrations of these forces adequate to meet any demand that may be made on the new doctrine. Now the tails of comets and the streamers of the corona, as explained by Arrhenius in Lick Observatory Bulletin 58, give abundant evidence of the operation of repulsive forces directed from the sun; but every case does not show a repulsion sufficiently strong to carry the particles away from our solar system on parabolic or hyperbolic paths. The question thus arises: Are there any known cases of repulsion sufficiently powerful to carry particles away from our sun to the other stars, and thus cause a secular decrease in the sun's mass? We may answer this question in the affirmative, for the following reasons:

1. Those comets which have had a very small perihelion distance, as the great comets of 1680, 1843 and 1882, have all had also very straight tails, which were found by calculation to be of immense length near perihelion passage. It is well known that this extreme straightness of tail indicates very powerful repulsion of the particles composing it.

2. By actual calculation I have established the fact that the velocity of the particles in the tails of the above comets, at perihelion, exceeded the parabolic velocity of a body driven away from the sun. The matter in these tails therefore was not only diffused over the solar system, but also carried away to other fixed stars.

3. Now if this repulsion with more than parabolic velocity could happen for vaporous matter developing in a comet's tail near perihelion, but remaining of sufficient density and luminosity to be visible

to the eye against the background of the sky, because it is condensed into a beam, the same thing obviously could develop also for particles in the solar corona itself, even if they be not sufficiently concentrated to present at night the aspect of a ray extending from the sun. In fact such rays of charged matter are proved to emanate from the sun by Maunder's researches on the sun spots and magnetic disturbances noted at Greenwich, and published in the *Monthly Notices* of the Royal Astronomical Society for 1905.

4. The emission of charged particles from the sun being thus clearly proved, the only question remaining open to discussion is whether any of the matter thus driven away from the sun goes away to the other fixed stars. But as my calculations show this to occur for the particles of the tails of comets which graze the sun's disc in perihelion—the only case in which the beams can be distinctly seen and the velocity of the particles determined from the lack of curvature in the tails—it must, by similarity of causes and effects, be held to occur also for some of the particles in the corona, even though they be invisible, owing to the diffuseness of the streamers.

5. The sun therefore is losing matter incessantly as well as gaining it, in the form of meteorites from celestial space. And in my "Researches," Vol. II., 1910, I have shown that the secular acceleration of the earth's motion indicates that at present the gain exceeds the loss; but if the sun was hotter in past ages, the reverse tendency formerly may have been at work.

6. Thus it appears to be demonstrated, by observed phenomena in our planetary system, that the sun is both gaining and losing matter, but that at present the rate of gain exceeds that of loss, so that there is a secular acceleration of the planets of such excessively minute character that it long escaped detection. In other fixed stars, it is probable that various combinations of gain and loss are at work; and we may be sure that the masses of the stars are not strictly constant over long ages, however approximately an even balance of gain and loss may hold for shorter intervals of time.

The view held by Newton and adopted by Lagrange and Laplace that the sun's mass may be considered constant, is only approximately true, and cannot properly be applied to the secular equations for the motions of the planets; and what has been found true of our sun, as respects a growth of mass, from the records of ancient eclipses, will naturally be adopted for other solar stars, while a secular decrease in mass may be assumed for some of the Sirian stars, owing to the intensity of their radiation.

XV. The Building of Clusters and Nebulæ Condensed Towards the Center, as Illustrated by the Very Elongated Elliptic Orbits of our System of Comets.

If we seek to inquire how clusters and nebulæ much condensed towards the center are built up by the process of capture, we shall find the general mathematical treatment by Green's theorem already given very satisfactory, for large bodies of the type of stars. It is equally convincing mathematically as applied to small bodies of the type of comets, but it is perhaps well to notice how the comets descending to our sun in very elongated ellipses have served to supply material for building up the planets and sun. This remarkable system of comets, with elliptic orbits equally diffused in all directions about our sun, is a sure sign that the nebulosity now condensed into our comets came originally from the fixed stars.

But if on the one hand, this equality of distribution of the aphelia in every direction points to the original entrance of the material into our nebula from without, the other equally remarkable property of high eccentricity, on the other, points to a similar conclusion. At the same time this coming in of matter from a distance makes possible the growth of the planets near the center of the system, because near perihelion the comets often pass so close to the planets as to have their orbits transformed, and their masses disintegrated and their dust absorbed by the planets. It is by moving against the resistance due to comets, and meteor swarms that the planetary orbits have been rendered so perfectly circular that the Greeks believed that the Deity had chosen the circle for the paths of the planets, because the circle was held by the ancient geometers to be a perfect figure.

Now what takes place about our sun, in the solar cluster, may also take place, in other star clusters of the Milky Way. There are in every region systems of bodies corresponding to our comets; and as they travel in very elongated ellipses, they tend to build up the bodies near the centers about which they revolve. In this way there must be countless infinities of comets working in towards the centers of the globular clusters; and thus they build up the equality of the stars in these regions, while at the same time the increase of mass and the resistance to orbital motion thus arising tend to round up the cluster and give increasing density towards the center.

Thus the analogy of the comets revolving in very elongated orbits and being destroyed to build up the planets and the sun, will also hold in the building up of a cluster. Not only may mature stars be captured and adjusted to the average oscillation within a cluster, but also myriads of millions of comets; and it is in this way largely that the cluster augments in mass and density towards the center. This growth of central power in turn augments the condensation observed in the clusters, and tends still further to produce a secular decrease in volume; just as the planets are drawn nearer the sun by the increase of the sun's mass. The shrinkage in the volume of a cluster is thus analogous to the diminution of the dimensions of the primitive orbits of the planets. And just as the planets in time will fall into the sun, so also will the stars of a cluster eventually combine into one great central star and thus produce an Arcturus or a Canopus.

The study of the system of comets about the sun, and the way the planets have been built up near the center of the solar nebula, thus gives us much light on the central accumulations noted in globular clusters. The smaller masses drawn in from without tend to augment the central bodies of the system; and this growth of mass in turn produces a further condensation of the original group, whether it be a planetary system, or a globular cluster of the highest order of glory and magnificence.

XVI. The Projectile Forces which Set the Double and Multipple Stars Revolving in Their Orbits, Point to Origin in the Distance.

If we have a stellar system made up of several components, we may designate the masses of the individual stars by M_1, M_2, M_3 . . ., M_n. We shall first consider a binary star with masses M_1 and M_2. Then the moment of momentum of the components about the common center of gravity of the system will be

$$\frac{M_1}{M_1+M_2}\left(\frac{M_2\rho}{M_1+M_2}\right)^2\Omega\sqrt{1-e^2} + \frac{M_2}{M_1+M_2}\left(\frac{M_1\rho}{M_1+M_2}\right)^2\Omega\sqrt{1-e^2} = \frac{M_1M_2}{M_1+M_2}\rho^2\Omega\sqrt{1-e^2}, \qquad (50)$$

where e is the eccentricity of the orbit, and ρ the radius vector, and Ω the mean angular velocity in the orbit (cf. inaugural dissertation, "Die Entwickelung der Doppel-Stern Systeme," Berlin, 1892, p. 16).

When the other elements are unchanged, we find that the moment of momentum of the binary system decreases with the increase of the eccentricity. In case of a circular orbit, e vanishes, and Ω is constant. In the general equation of the planetary theory the unit of time may be so chosen that the constant of attraction (cf. Gauss, "Theoria Motus," Lib. I., § 1) becomes

$$k^2 = \frac{4\pi^2}{\tau^2}\cdot\frac{a^3}{M+m} = n^2a^3 = 1, \quad \text{or} \quad n = \frac{1}{a^{\frac{3}{2}}}; \qquad (51)$$

and we may therefore put Ω for n and ρ for a, and the second member of (50) becomes

$$\frac{M_1M_2}{M_1+M_2}\rho^2\Omega\sqrt{1-e^2} = \frac{M_1M_2}{M_1+M_2}\rho^{\frac{1}{2}}\sqrt{1-e^2}, \qquad (52)$$

the radical involving e to be unity in circular orbits.

From this equation (52) it appears that with constant mass the moment of momentum of a system of double stars depends on the square root of the mean radius vector, and therefore increases rapidly with the distance.

Other conditions being equal, the maximum moment of momentum would therefore be attained by the separation of two stars to a great distance, yet a pair of such passing stars would have to have peculiar directions and velocities to enable them under mutual gravitation to form a system. If the motions of two stars were directed towards the same point in space, and with velocities which would enable one to overtake the other, before or after the point was reached, one might revolve about the other; and with proper relative velocity—to be gotten either by altering the directions of motion, or by adjusting the velocities in the converging lines of motion—the two stars might form a binary system.

This dynamical condition of formation is so difficult to realize in practice that we may be sure that it is quite rare in nature; and that the vast majority of double stars have developed from nebulæ, by the appropriate division of the elements between two leading centers of condensation. But it is now recognized that the nebulæ themselves have developed from dust expelled from the fixed stars and were originally of vast extent; and hence even if the bodies into which they condense gradually approach the center of gravity of the system, as the stars increase in mass and revolve against the nebular resisting medium and their orbits grow smaller and smaller and rounder and rounder, it will yet follow that many double stars have components so far apart that their systems have large moments of momentum of orbital motion.

The difficulty of explaining the large orbital moments of momentum of double stars first arose in completing certain calculations for my inaugural dissertation at the University of Berlin just twenty years ago. At that time I saw that a wide separation of the components of a system gave large moment of momentum, and that in order to account for the orbital moment of momentum by the hypothesis of tidal friction first developed by Sir George Darwin and afterwards extended by me to binary systems, it was necessary to endow the stars with very rapid axial rotation. Otherwise the mean distance of the components would not be greatly increased by the exhaustion of the moments of momentum of axial rotation under the secular action of tidal friction.

At this early stage in the study of the problems of cosmogony, naturally I had not exhausted the other possible modes of formation, though I had largely excluded the capture of single stars by chance approach due to difference in proper motion. The further study of this problem has occupied a part of the past twenty years, but as it has now led to the establishment of a great law of nature, one may feel that the labor has not been in vain.

From the above reasoning it will be found:

1. That if the globes of the stars of a binary be expanded till a hydrostatic connection is established between the components, the fluid will thereby become so rare that no hydrostatic pressure could be exerted to throw off a companion by rotation.

2. A rotation rapid enough to produce such a separation could not be accounted for by natural causes.

3. Hence it is clear that the premise implying a separation by rotation is false; and the true mode of formation is diametrically opposite to what was long believed. Instead of being thrown off by rapid rotation, the attendant bodies have all been formed in the distance, and added on from without, so that they have neared the centers about which they now revolve. This uniform law greatly simplifies all our conceptions of cosmical evolution.

To illustrate the relative significance of the moments of momentum of the axial rotations compared to the moment of momentum of orbital motion, it suffices to cite the case considered in my inaugural dissertation of 1892, pp. 37–38. In this case each of the two equal stars imagined expanded into a nebula has three times the mass of the sun; and the axial rotations are such as to give an oblateness of 2/5. The stars are set in motion at a mean distance of 30 astronomical units. In the special units there adopted, it turns out that the moments of momentum of the axial rotations have the numerical values 0.394, or 0.788 for the two stars; and the moment of momentum of orbital motion becomes 2.378.

Thus with the two stars so far apart as 30, it is impossible to keep the figures of equilibrium stable and yet give them rotations rapid enough to render the moments of momentum of axial rotation large compared to that of the orbital motion. Nevertheless, a double star orbit with a mean distance of 30 must be considered small compared to many orbits which exist in the heavens. For there are physically connected stars which show very little motion in a century, and others which remain quite fixed, as may be clearly established by comparing modern measures with those of Herschel and Struve.

The conclusion from this calculation is that the observed mean distance of wide double stars has not been developed by the transfer of moment of momentum of axial rotation to moment of momentum of orbital motion. By such transfer of moment of momentum the orbit may indeed be expanded, but not to many times its original size. On this tidal frictional theory the larger orbits of double stars could not be explained satisfactorily. The difficulty encountered some twenty years was therefore first overcome in developing the second volume of my "Researches," along the lines of thought resulting from the extension of Babinet's criterion in 1908.

Looking at the problem in the light of recent progress it is evident that the large and highly eccentric orbits of double stars do undoubtedly point to capture; that is, the formation of separate nuclei at a great distance, and the revolution of the two stars in narrowing orbits about the center of gravity of the system. If this process of revolution in the original nebula should continue long enough, the size and eccentricity of the orbit would be much reduced; and we should thus obtain systems of the type commonly observed to be in comparatively rapid revolution. There is thus established a real connection between the revolving visual double stars and the much larger number of physical systems which have remained nearly if not quite fixed since the epoch of Herschel and Struve.

This inference is also sustained by recent progress in double star astronomy, which shows that the longer the period the higher the eccentricity, and the same tendency holds for the rapid spectroscopic binaries, as I pointed out in 1907 (*Monthly Notices*, Roy. Astron. Soc., Nov., 1907). This unbroken continuity among all the classes of double stars shows that the cause is everywhere the same. If therefore the wider visual double stars have formed from separate nuclei, in the condensing nebulæ, the explanation becomes valid also for the spectroscopic binaries; and the law of formation is the same for all the double stars as for the planets of the solar system, where Babinet's criterion is absolutely decisive against the detachment theory generally held since the days of Laplace, but now universally abandoned.

Summary and Conclusions.

Without attempting in this closing summary to recapitulate the contents of this memoir in detail, it may yet be well to draw attention to some of the most significant conclusions at which we have arrived.

1. As intimated in the first section of this paper the problem of n-bodies, under ideal dynamical conditions, remains forever beyond the power of the most general methods of analysis; but the dynamical theory of clusters gives us the one secular solution of this problem found under actual conditions in nature. For when n is of the order of 1,000, so as to give rise to a cluster, the clustering power observed by Herschel operates to exhaust the mutual potential energy of the system, and bring about increasing accumulation in the center, so that the cluster finally unites into a single mass of enormous magnitude. Probably the giant stars of the type of Canopus and Arcturus have arisen in this way.

2. And since attendant bodies of every class—as satellites, planets, comets, double and multiple stars—tend everywhere to approach the centers about which they revolve, as an inevitable effect of the growth of the central masses and of the action of the resisting medium over long ages, it follows that the secular solution of the problem of clusters is more or less valid for all cosmical systems. They finally end by the absorption of the attendant bodies in the central masses which now govern their motions.

3. The dynamical theory of globular clusters shows that the clustering power inferred by Herschel is nothing else than the action of universal gravitation; and that it operates on all sidereal systems, but does not produce the cumulative effect which Herschel ascribed to the ravages of time inside of millions of ages.

4. The globular clusters are formed by the gathering together of stars and elements of nebulosity from all directions in space; and

this points to the expulsion of dust from the stars of the Milky Way, and its collection about the region of the formation in such manner as to give essential symmetry in the final arrangement of the cluster, which doubtless has some motion of rotation, and originally a tendency to spiral movement.

5. The stars and smaller masses are captured by the mutual action of the other members of the cluster, and worked down towards the center of the mass. This gives a central density in excess of that appropriate to a sphere of monatomic gas in convective equilibrium (*A. N.*, 4053, and *A. N.*, 4104).

6. The density of the clusters is greater on the outer border than in a globe of monatomic gas, which shows that stars are still collecting from the surrounding regions of space. The starless aspect of the remoter regions about clusters is an effect of the ravages of time, as correctly inferred by Herschel in the course of his penetrating sweeps of the starry heavens.

7. And just as clusters under the mutual gravitation of the component stars contract their dimensions, with time, chiefly owing to the growth of the central masses, so also do other systems, whether the mass-distribution be *single,* giving a system made up of a sun and planets, or *double, triple* and *multiple,* giving binary, triple or multiple stars, or sidereal systems of still higher order. The tendency everywhere is from a wider to a narrower distribution of the large bodies; while the only throwing off that ever occurs is of particles driven away from the stars by the action of repulsive forces.

8. The orbits of the stellar and planetary systems are decreased by the growth of the central masses and rounded up by the action of the nebular resisting medium. And in like manner all clusters tend to assume spherical or globular figures, so as to justify the expression of Plato, that the Deity always geometrizes; or Newton's remark that the agency operating in the construction of the solar system was "very well skilled in mechanics and geometry."

9. Newton required the intervention of the Deity to give the planets revolving motion in their orbits, because in the absence of repulsive forces he could not account for the dispersion of the matter, so as to produce the tangential motions actually observed. By means

of the theory of repulsive forces, however, it is now possible to explain these projectile motions, which Herschel likewise pointed to as the chief agency for the preservation of sidereal systems. The only assumption necessary is an unsymmetrical figure of the primordial nebula, giving a whirling motion about the center as the system develops; and since the dust gathers from all directions it is certain that this lack of perfect symmetry will nearly always develop, as we see also by the spiral nebulæ.

10. It is this unsymmetrical form of the spiral nebulæ produced by the gathering of the dust from the stars, or the slight relative tangential motion of stars formed separately but finally made to revolve together as a binary system, that gives the binary stars the projectile forces, with which they are set revolving in their orbits. In no case have they resulted from the rupture of a rotating mass of fluid under conditions of hydrostatic pressure as formerly believed by Darwin, Poincaré and See.

11. Even if the rotation could become rapid enough to produce a separation, under conditions of hydrostatic pressure, by rupture of a figure of equilibrium, there would still be the equal or greater difficulty of explaining the origin of the primitive rapid rotation. This last difficulty escaped notice till we came to assign the cause of rotations, and found that mechanical throwing off was impossible under actual conditions in nature. It is therefore recognized, from the definite proof furnished by Babinet's criterion in the solar system, that such a thing as a throwing off never takes place; but that all planetary and stellar bodies are formed in the distance, and afterwards near the centers about which they subsequently revolve.

12. *This gives us a fundamental law of the firmament—the planets being added on to the sun, the satellites added on to their planets, the moon added on to the earth, and the companions added on to the double and multiple stars—which is now found to be beautifully confirmed by the dynamical theory of the globular clusters. It is not often that such a great law of nature can be brought to light, and it is worthy of the more consideration from the circumstance that it explains all classes of stellar systems by a single general principle.*

13. As sidereal systems of lower order are conserved by projectile forces, it is probable that the clusters likewise have a spiral motion of rotation, with similar projectile forces tending to counteract simple progressive collapse. The period of the orbital revolution of the stars of a cluster is found to be common to all, without regard to the dimensions of the elliptical orbits described, and thus the whole system may have a common period of oscillation, after which the initial condition is perfectly restored. This possibility in the dynamics of a cluster is exceedingly wonderful, and results from the central attraction depending directly on the distance.

14. The equality of brightness in star clusters shows that some process of compensation between the attractive and repulsive forces has produced stars of wonderful uniformity of luster. Thus the present investigation confirms the previous researches on the evolution of the stellar systems, which have laid the foundations for a new science of the starry heavens.

15. Accordingly the capture theory of cosmical evolution being now firmly established for the clusters, where the nature of the process is entirely clear, it becomes at once a guide to us in dealing with systems of lower order; and we see that the law of nature is uniform and everywhere the same, the large bodies working in towards the centers of attraction, while the only throwing off that ever takes place is of small particles driven out of the stars by the action of repulsive forces. All planetary bodies are formed in the distance, and have their orbits reduced in size by increase of the central masses, and rounded up by moving in a resisting medium. *This is a perfectly general law of the sidereal universe. It verifies the early conjectures of Plato and Newton as to the stability of the order of the world, and shows that these illustrious philosophers were quite justified in concluding that the Deity always geometrizes.* The spiral nebulæ tend to develop systems with rounder and rounder orbits, and the clusters made up of thousands of stars assume globular figures with minimal surfaces and internal density so arranged as to give maximum exhaustion of the potential energy.

16. This is geometry of the most marvellous kind, as we find it impressed on the systems of the sidereal universe; and the perfection

of this most beautiful science of celestial geometry may be considered the ultimate object of the labors of the astronomer. The philosophic observer is not and never can be content with mere observations of details which do not disclose the living, all-pervading spirit of nature.

17. If, then, the mystery of the gathering of stars into clusters is now penetrated and traced to the clustering power of universal gravitation, so also is the mystery of the *converse problem of starless space,* which was a subject of such profound mediation by the great Herschel.

18. This incomparable astronomer likewise correctly concluded that the breaking up of the Milky Way into a *clustering stream* is an inevitable effect of the ravages of time; but we are now enabled to foresee the restorative process, under the repulsive forces of nature, by which new nebulæ, clusters and sidereal systems of high order eventually will develop in the present depopulated regions of starless space.

19. If there be an incessant expulsion of dust from the stars to form the nebulæ, with the condensation of the nebulæ into stars and stellar systems, while the gathering of stars drawn together by a clustering power operating over millions of ages gives at length a globular mass of thousands of stars accumulating to a perfect blaze of starlight in the center, but surrounded externally by a desert of starless space resulting from the ravages of time, certainly the building of these magnificent sidereal systems may well engage the attention of the natural philosopher.

20. The foremost geometers of the eighteenth century, including Lagrange, Laplace and Poisson, were greatly occupied with the problem of the stability of the solar system; and in his historical eulogy on Laplace the penetrating Fourier justly remarks that the researches of geometers prove that the law of gravitation itself operates as a preservative power, and renders all disorder impossible, so that no object is more worthy of the meditation of philosophers than the problem of the stability of these great celestial phenomena.

But if the question of the stability of our single planetary system *may so largely absorb the talents of the most illustrious geometers of the age of Herschel, how much more justly may the problem of the stability of clusters, involving many thousands of such systems, claim the attention of the modern geometer, who has witnessed the perfect unfolding of the grand phenomena first discovered by that unrivaled explorer of the heavens?*

The grandeur of the study of the origin of the greatest of sidereal systems is worthy of the philosophic penetration of a Herschel! The solution of the dynamical problem presented surpasses the powers of the most titanic geometers, and would demand the inventive genius of a Newton or an Archimedes!

Yet notwithstanding the transcendent character of the problem, and the hopelessness of a rigorous solution in our time, even an imperfect outline of nature's laws may aid the thoughtful astronomer, in penetrating the underlying workings of the sidereal universe, and thus enable him to perceive the great end subserved by the development of the cosmos. If so, he may well rejoice, and exclaim with Ptolemy:

> "Though but the being of a day,
> When I the planet-paths survey,
> My feet the dust despise;
> Up to the throne of God I mount
> And quaff from an immortal fount
> The nectar of the skies."

Starlight on Loutre,
Montgomery City, Missouri,
February 19, 1912.